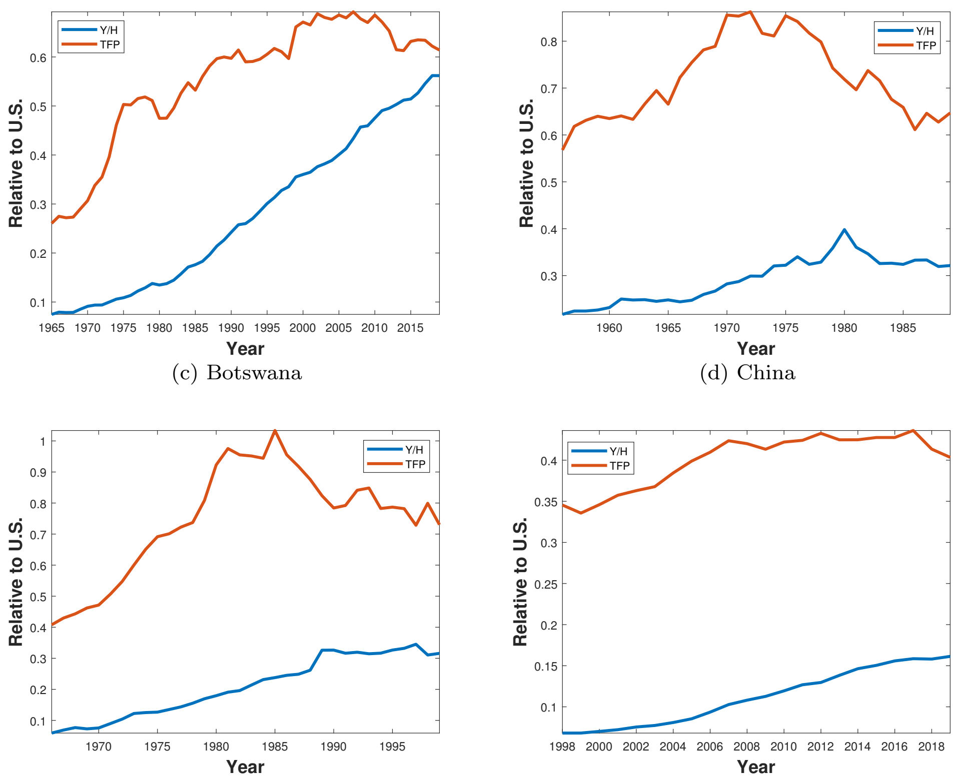
1 Technological adoption: Catching-up to the frontier

In the Romer model, we have studied an economy like the U.S. where researchers discover entirely new ideas and push the technological frontier forward in a steady way. We have seen already in the data that, in the case of Korea, a country that is far away from the technological frontier can have much faster growth rates of TFP. Figure 1 highlights that phenomena for three further countries that are considered growth miracles: Brazil from 1950-1980, Botswana from 1960-1990, and China from the end of the 1990s to today. All those countries substantially closed over that period their output per hour gap with the U.S., and all countries experiences a much higher TFP growth rate than the U.S.

Figure 1: TFP growth during growth miracles



(a) Korea

(b) Brazil

(c) Botswana

(d) China

It appears natural that such rapid technological change should be possible.

Once an idea is developed, adopting it should be much easier than inventing it completely anew. For example, building a nuclear power plant using designs from the 1990s and 2000s takes about 11 years, and countries like Turkey, Bangladesh, and China are all adopting the technology. However, developing the latest technology of nuclear reactors, the so called fourth generation, has taken already more than 20 years.

Despite technological growth being fast in the examples above, it still took these countries decades to meaningfully reduce the gap in TFP to the U.S. What is more, non of the countries, except Botswana for a short period of time, fully closed the gap to the U.S. Moreover, we know that large TFP differences across countries are persistent over time. All these observations suggest that adopting newer technologies cannot be frictionless.

Here, we are going to see two frictions. First, we are going to consider human capital. The idea is that a more skilled workforce can operate a wider range of capital goods that are available from the technological frontier. In doing so, we are going to maintain the assumption of the representative household, i.e., we measure the average skill level of an economy. Obviously, some of the brightest Chinese have studied at the best universities in the U.S. and are just as capable of using advanced capital goods as the brightest U.S. students. This alone, however, does not necessarily provide West-Chinese farmers access to these capital goods. Second, we are going to enrich the model by trade that provides a country with capital goods that are beyond those its skill level alone would allow for. There are several ways to think about the role of trade on the availability of capital goods: The simplest is that the level of skills determining the number of capital-goods varieties that can be produced at home, and trade allows a country to import additional varieties. Using the example from above, Bangladesh is importing the technology for its nuclear reactor from Russia. A more indirect channel may be that by engaging with foreign firms, through importing and exporting, domestic firms learn production processes.

Throughout, we will retain the assumption that one country is able to use a strict subset of technologies from the technological frontier. Again, this is a simplification from reality. For example, Chinese companies have adopted the fast-fashion technology developed by European companies such as *Zara* during the 1980s, however, has partly surpassed their technology with companies such as *Shein* that offers thousands of new designs daily and has a design to market time of less than a week. Similarly, *Nvidia* is producing some of its most advanced computer chips in Taiwan. At the same time, Taiwan, and even more China, have large shares of their population using much less advanced technologies than the average U.S. worker, and it is the average we wish to understand.

1.1 Model set up

The final goods production process employs the insights from the Romer model. Output is produced using $h(t)$ differentiated capital goods and labor:

$$Y(t) = L(t)^{1-\alpha} \int_0^{h(t)} x_j(t)^\alpha dj. \tag{1}$$

Here, $h(t)$ is the measure of capital goods that the country knows how to use in period t. This will be different from the technological frontier $A(t)$. As in the Romer model, each unit of capital goods needs to be produced by an intermediate good producer using a unit of the aggregate available aggregate capital stock:

$$\int_0^{h(t)} x_j(t)dj = K(t). \tag{2}$$

As all goods are equally productive and will charge the same price, each unit is used in the same proportion:

$$x_j(t) = x(t) = \frac{K(t)}{h(t)}. \tag{3}$$

Substituting the result into the production function yields

$$Y(t) = L(t)^{1-\alpha} h(t) x(t)^\alpha \tag{4}$$

$$Y(t) = \left(L(t)h(t)\right)^{1-\alpha} K(t)^\alpha. \tag{5}$$

Different from the Romer model, we abstract from an explicit research sector. Instead, we assume that all labor is employed in the final good sector. The wage in that sector and the rental price of capital are given by

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = 1 - \alpha \frac{Y(t)}{L(t)} \tag{6}$$

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha \frac{Y(t)}{K(t)}. \tag{7}$$

Hence, we have again for household income that it equals otal output:

$$\tilde{Y}(t) = w(t)L(t) + r(t)K(t) = Y(t). \tag{8}$$

As always, this result simplifies for us to think about the aggregate capital accumulation from the household which is given by our familiar law of motion:

$$\dot{K}(t) = sY(t) - \delta K(t), \tag{9}$$

Moreover population grows at an exogenous rate n:

$$\dot{L}(t) = nL(t). \tag{10}$$

Next, we turn to the dynamic equation for the accumulation of the skill level, $h(t)$. As eluded to above, we abstract from an explicit research sector that copies the ideas from the frontier. Instead, we assume that the mass of differentiated capital goods that we learn to use, $\dot{h}(t)$, depends on the skill level of the workforce:

$$\dot{h}(t) = \mu \exp(\psi u) A(t)^{\gamma} h(t)^{1-\gamma}, \tag{11}$$

where u measures the time people spend learning new skills, ψ is its return, and μ measures the quality the learning environment. One way to think about the skill investment is as we did in the Solow model, namely formal education. However, it may also encompass more informal ways of learning skills such as learning-by-doing. Again using the insights from the Romer model, we will assume that the current level of skills makes it easier to learn new skills, i.e., $\gamma < 1$. Moreover, we will assume that a further developed technological frontier makes it easier to lean new skills, i.e, $\gamma > 0$. Put differently, the further the economy is from the technological frontier, the easier it becomes to adopt new ideas. This becomes yet more explicit when we write the equation in growth rates:

$$\frac{\dot{h}(t)}{h(t)} = \mu \exp(\psi u) \left(\frac{A(t)}{h(t)} \right)^{\gamma} \tag{12}$$

The growth rate is proportional to the distance to the technological frontier. Finally, we assume that the frontier grows according to

$$\frac{\dot{A}(t)}{A(t)} = g. \tag{13}$$

1.2 Steady state

Given the production function, as well as the familiar laws of motion for labor and capital, we know that the capital-to-output ratio solves

$$z^* = \left(\frac{K(t)}{Y(t)} \right)^* = \frac{s}{n + g_h(t) + \delta}, \tag{14}$$

which has a steady state if the growth rates is a constant, i.e., $g_h(t) = g_h$. Moreover, we know that we can write output per worker in steady state as

$$\left(\frac{Y(t)}{L(t)} \right)^* = \left(\frac{s}{n + g_h + \delta} \right)^{\frac{\alpha}{1-\alpha}} h(t), \tag{15}$$

where the only time-varying variable is $h(t)$. We will first derive g_h, i.e., show that a steady state capital-to-output ratio exists and then find a solution for $h(t)$.

Finding the steady state growth rate of skills turns out to be just as easy as in the Romer model. For the growth rate of skills

$$\frac{\dot{h}(t)}{h(t)} = \mu \exp(\psi u) \left(\frac{A(t)}{h(t)} \right)^{\gamma} \tag{16}$$

to be constant, we directly see that $\frac{A(t)}{h(t)}$ needs to be constant which implies $g_h = g$, i.e., in steady state, skills grow at the rate of the technological frontier.

Once we have the growth rate of skills in steady state, it is easy to see the growth rate of output per worker in steady state. In Equation (15), $h(t)$ is the only time-varying variable. Hence, the growth rate of output per worker in steady state is also g.

At first though this may sound like an undesirable feature: should we not expect a country to converge to the technological frontier and the associated output per worker as long as it is strictly below? Considering again the cases above makes clear that it is a desirable feature: Brazil, Botswana, China, and Korea ultimately reached TFP levels that remained persistently below the level of the U.S. Moreover, among these countries, China and Korea are the only countries that kept converging in terms of output per worker relative to the U.S. and this convergence was not driven by convergence in technology but by an increasing capital-to-output ratio or workers becoming more educated. We can use our model to see what parameters determine the distance to the frontier in steady state:

$$\frac{\dot{h}(t)}{h(t)} = \mu \exp(\psi u) \left(\frac{A(t)}{h(t)}\right)^{\gamma} \tag{17}$$

$$g = \mu \exp(\psi u) \left(\frac{A(t)}{h(t)}\right)^{\gamma} \tag{18}$$

$$\frac{h(t)}{A(t)} = \left(\frac{\mu \exp(\psi u)}{g}\right)^{\frac{1}{\gamma}}. \tag{19}$$

One directly sees that $\mu \exp(\psi u)$ increases the ratio, i.e., an environment more conducive to skill accumulation pushes the economy closer to the technological frontier. Note, g decreases the ratio. A faster growth at the frontier makes it harder for the less developed economy to keep up.

Next, we need to obtain a solution for the skill level $h(t)$. To that end, we need to solve the differential equation of skill accumulation:

$$\dot{h}(t) = \mu \exp(\psi u) A(t)^{\gamma} h(t)^{1-\gamma}. \tag{20}$$

We have to deal with the non-linearity in $h(t)$. To that end, define the auxilary variable

$$v(t) = h(t)^{\gamma}, \tag{21}$$

Taking the derivative with respect to time and applying the chain rule gives us

$$\dot{v}(t) = \dot{h}(t)\gamma h(t)^{\gamma-1}. \tag{22}$$

Substituting yields

$$\dot{v}(t) = \gamma\mu \exp(\psi u) A(t)^{\gamma}. \tag{23}$$

Note, the only variable depending on time is $A(t)$. To make that explicit, substitute $A(t) = A(0)\exp(gt)$:

$$\dot{v}(t) = \gamma\mu \exp(\psi u) A(0)^{\gamma} \exp(\gamma g t). \tag{24}$$

the solution to the differential equation is

$$v(t) = \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \exp(\gamma g t) + l, \tag{25}$$

where l is the integration constant. To obtain an economic interpretation, evaluate the equation at $v(0)$:

$$l = v(0) - \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \tag{26}$$

Plugging in and substituting again $v(t) = h(t)^{\gamma}$ yields:

$$h(t)^{\gamma} = \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \exp(\gamma g t) + h(0)^{\gamma} - \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \tag{27}$$

$$h(t) = \left(h(0)^{\gamma} + \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \left[\exp(\gamma g t) - 1\right] \right)^{\frac{1}{\gamma}}. \tag{28}$$

Little surprising, a higher initial skill level, $h(0)$, increases skills today. Similarly, a higher initial level of the technological frontier, $A(0)$ increases skills, as a higher level implies that adopting capital goods is simpler. Finally, a more conducive skill accumulation environment, $\mu \exp(\psi u)$, increases the skill level. Naturally, this latter effect depends on the technological frontier, as the importance of that environment for skill accumulation interacts with the technological frontier. Put differently, having a good skill accumulation environment becomes the more important for skill accumulation, when the technological frontier is very advanced.

Plugging in the solutions of g_h and $h(t)$ into the steady state equation of ourput per worker yields

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g_h+\delta}\right)^{\frac{\alpha}{1-\alpha}} \left(h(0)^{\gamma} + \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \left[\exp(\gamma g t) - 1\right]\right)^{\frac{1}{\gamma}}. \tag{29}$$

There are several observations worth highlighting. First, a higher capital to output ratio leads to higher output per worker. As the former is decreasing in n, for a developing economy, population growth decreases output per worker unambiguously as it no longer affects the number of capital goods that the economy uses. This makes the model consistent with the cross-sectional evidence we discussed in the context of the Solow model that showed a negative relationship between output per worker and the population growth rate. Second, the model provides a reason why skills lead to higher output per worker. Skills do not make workers more productive in the abstract, as in the Solow model with education, but they make workers more productive by allowing them to use more capital goods. Third, pushing out the technological frontier benefits all countries as they can copy those new ideas over time.

1.3 Out of steady state dynamics

The above examples highlight that temporary fast technological growth, that leads to temporary fast output per worker growth, is possible. Our model explains such rapid growth through convergence dynamics to the steady state. We can use the solution for the level of skills to solve for these transition dynamics:

$$\frac{\dot{h}(t)}{h(t)} = \mu \exp(\psi u) \left(\frac{A(t)}{h(t)}\right)^{\gamma} \tag{30}$$

$$h(t) = \left(h(0)^{\gamma} + \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma} \left[\exp(\gamma g t) - 1\right]\right)^{\frac{1}{\gamma}} \tag{31}$$

Combining the equations and rewriting gives:

$$
\begin{aligned}
\frac{\dot{h}(t)}{h(t)} &= \mu \exp(\psi u) \frac{A(t)^{\gamma}}{h(0)^{\gamma} + \frac{1}{g}\mu \exp(\psi u) A(t)^{\gamma} - \frac{1}{g}\mu \exp(\psi u) A(0)^{\gamma}} && (32) \\
&= g \frac{\mu \exp(\psi u) A(t)^{\gamma}}{h(0)^{\gamma} g + \mu \exp(\psi u) A(t)^{\gamma} - \mu \exp(\psi u) A(0)^{\gamma}} && (33) \\
&= \frac{g}{1 + \left(\frac{h(0)}{A(t)}\right)^{\gamma} \frac{g}{\mu \exp(\psi u)} - \left(\frac{A(0)}{A(t)}\right)^{\gamma}}. && (34)
\end{aligned}
$$

Several observations are worth making. As always, we first may want to make sure we did not make a mistake, i.e., it should always be true that when we start in steady state, we obtain the steady state growth rate. This is indeed the case: When $\frac{h(0)}{A(0)} = \left(\frac{\mu \exp(\psi u)}{g}\right)^{1/\gamma}$ and $A(0) = A(t)$, then $\frac{\dot{h}(t)}{h(t)} = g$. Economically more interesting, the equation shows that the lower is $\frac{h(0)}{A(0)}$ relative to its steady state, the faster is the initial growth rate. That is, an economy that is particularly far away from its new steady state should have particular high technological growth rates. However, over time, as time passes, $A(t)$ grows, and $\frac{\dot{h}(t)}{h(t)} \mapsto g$. Taken together, we have again convergence to the steady state from any initial point, and when starting below the steady state, the growth rate should slow down over time until it converges to its steady state. Finally, for any initial values of $h(0), A(0)$, increasing μ, ψ, u increases the growth rate of skills. That is, as seen above, a more conducive skill learning environment leads to a permanently higher skill level which will lead to a temporary increase in the growth rate of skills along the transition path.

The role of growth at the frontier, g, on the growth rate is not directly obvious. On the one hand, it increases the steady state growth rate. On the other hand, as discussed above, it increases the steady state gap between the country's skills and the technological frontier. To see which effect dominates, consider

$$
\frac{\partial \frac{\dot{h}(t)}{h(t)}}{\partial g} = \frac{1 - \left(\frac{A(0)}{A(t)}\right)^{\gamma}}{\left[1 + \left(\frac{h(0)}{A(t)}\right)^{\gamma} \frac{g}{\mu \exp(\psi u)} - \left(\frac{A(0)}{A(t)}\right)^{\gamma}\right]^2}. \qquad (35)
$$

Under our assumption of $g > 0$, the fraction is always positive, i.e., faster growth at the frontier increases the growth rate of skills along the entire transition path.

Having understood the transition dynamics for skills, we can now turn to output per worker. As before, let us write output per worker as a function of the capital to output ratioÑ

$$Y(t) = h(t)L(t)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \tag{36}$$

$$y(t) = h(t)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \tag{37}$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{h}(t)}{h(t)} + \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)}. \tag{38}$$

As always, the dynamics of the capital-output-ratio are given by

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{s}{z(t)} - (1-\alpha)\left(n + \frac{\dot{h}(t)}{h(t)} + \delta\right). \tag{39}$$

Combining the equations gives

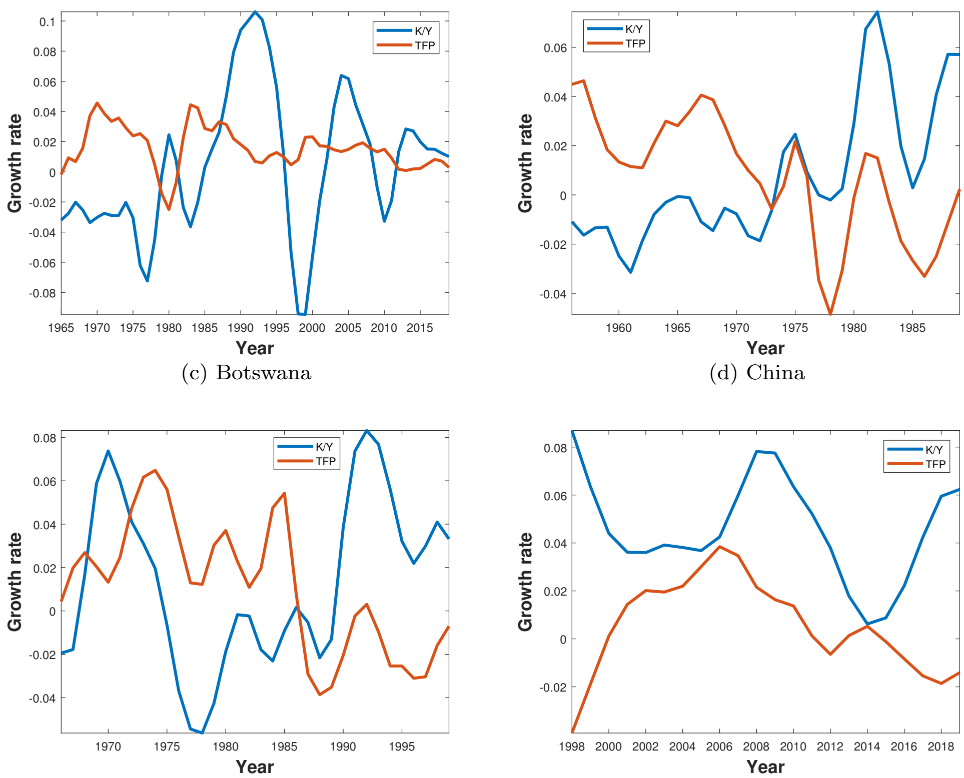
$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{h}(t)}{h(t)} + \alpha\left[\frac{s}{z(t)} - \left(n + \frac{\dot{h}(t)}{h(t)} + \delta\right)\right]. \tag{40}$$

The equation highlights that, similar to the Romer model, a temporary increase in the growth rate of skills increases the growth rate of $y(t)$ but it increases it initially by less than the increase in the growth rate of skills because the capital to output ratio declines initially. Figure 2 shows that this prediction is qualitatively in line with the data from our four growth miracles. During the time when TFP was growing fastest (Korea in the late 60s and early 80s, Brazil in the 50s and 60s, Botswana in the 70s, and China in the early 2000s), the capital-to-output ratio was growing relatively slowly.

1.4 Introducing trade

So far, we assume that acquiring the knowledge to use more advanced capital goods is the only way to obtain those. However, there are good reasons to believe that trade is a way for new technologies becoming available to countries. Modern China

Figure 2: TFP growth and the capital-to-output ratio



(a) Korea (b) Brazil

(c) Botswana (d) China

is a good example. China imports major capital goods from Europe and the U.S. used in the production process. For example, the recent advances in AI coming from China would not have been possible without having imported previously advanced chips and codes. Trade is particularly important in the presence of international patenting which assures that simply copying foreign capital goods is not possible.

We also have systematic data support that trade can lead to productivity growth. López and de Blas Pérez (2018) study trade in manufacturing with the U.S. for several European countries. They find that trade with the U.S. lowers the costs of capital goods, i.e., the available capital goods become more productive, increases investment, and increase output (see Table 3).

Figure 3: Trade and production

Table 3: Bilateral trade linkages and the European MTC

	GDP per working age population			
Bilateral exports US	France	Germany	Italy	Spain
Lag 0	0.6509*	0.7485*	0.7825*	0.7994*
Lag 1	0.7058*	0.7670*	0.7938*	0.7640*
Lag 2	0.7079*	0.7521*	0.7643*	0.6943*
Lag 3	0.6798*	0.7136*	0.7066*	0.6091*
	Investment per working age population			
Bilateral exports US	France	Germany	Italy	Spain
Lag 0	0.6150*	0.7905*	0.4726*	0.7965*
Lag 1	0.6222*	0.7340*	0.4917*	0.7770*
Lag 2	0.5733*	0.6283*	0.4702*	0.7184*
Lag 3	0.5001*	0.5012*	0.4287*	0.6504*
	Relative price of capital			
Bilateral exports US	France	Germany	Italy	Spain
Lag 0	-0.6575*	-0.1974*	-0.7331*	-0.5444*
Lag 1	-0.6416*	-0.2554*	-0.6604*	-0.4967*
Lag 2	-0.6218*	-0.3134*	-0.5667*	-0.4378*
Lag 3	-0.5838*	-0.3694*	-0.4522*	-0.3617*

Source: López and de Blas Pérez (2018)

1.4.1 Model set up

We augment the above production function by imported capital goods. As before, let $h(t)$ be the amount of capital goods that are produced in a country. Moreover, let $m(t)$ be the imported goods. Than output is

$$Y(t) = L(t)^{1-\alpha} \int_0^{h(t)+m(t)} x_j(t)^\alpha dj. \tag{41}$$

As before, given the assumption of homogeneity of capital goods, it is optimal to produce the same amount of each capital good. Let this amount be $z(t)$. Hence, capital markets clearing implies A country produces $z(t)$ units of each home-based capital good and, hence,:

$$z(t)h(t) = K(t). \tag{42}$$

Turning to trade, given the production function, the country optimally uses the same amount of each capital good, both home produced and imported. Let this amount be $x(t)$, i.e., the country keeps only $x(t)h(t)$ of the home-produced goods with $x(t) < z(t)$ and buys $x(t)$ of the imported goods. Hence, balanced trade implies:

$$x(t)m(t) = K(t) - x(t)h(t) \tag{43}$$

$$K(t) = x(t)[m(t) + h(t)]. \tag{44}$$

Given these results, we can rewrite the production function as

$$Y(t) = L(t)^{1-\alpha} \int_0^{h(t)+m(t)} x_j(t)^\alpha dj \tag{45}$$

$$Y(t) = L(t)^{1-\alpha}(h(t) + m(t))x(t)^\alpha. \tag{46}$$

Using the trade balance, we have

$$Y(t) = L(t)^{1-\alpha}(h(t) + m(t)) \left(\frac{K(t)}{m(t) + h(t)}\right)^\alpha \tag{47}$$

$$Y(t) = ([m(t) + h(t)]L(t))^{1-\alpha} K(t)^\alpha. \tag{48}$$

1.4.2 The steady state

To find a steady state, define the total number of capital goods as $N(t) = h(t) + m(t)$. Next, consider again the two equations determining the dynamics of the capital-tooutput ratio: The growth rate of the capital-to-output ratio, and the capital accumulation equation:

$$z(t) = \frac{K(t)^{1-\alpha}}{(N(t)L(t))^{1-\alpha}} \qquad \frac{\dot{z}(t)}{z(t)} = 1 - \alpha\frac{\dot{K}(t)}{K(t)} - 1 - \alpha \left(n + \frac{\dot{N}(t)}{N(t)}\right) \tag{49}$$

$$\frac{\dot{z}(t)}{z(t)} = \frac{s}{z(t)} - \delta. \tag{50}$$

Combining the equations and assuming that a steady state exists, we have

$$n + \frac{\dot{N}(t)}{N(t)} = \frac{s}{z^*} - \delta \tag{51}$$

$$z^* = \frac{s}{n + \frac{\dot{N}(t)}{N(t)} + \delta} \tag{52}$$

which is constant if $\frac{\dot{N}(t)}{N(t)} = g_N(t)$ is constant. To see that this is the case, let us derive the growth rate:

$$\ln n(t) = \ln\left(x(t) + m\right) \tag{53}$$

$$g_N(t) = \frac{\dot{x}(t)}{x(t) + m}. \tag{54}$$

Note, as $x(t)$ is growing at the rate of the technological frontier, g, in steady state, $x(t) + m \mapsto x(t)$ and $g_N(t) \mapsto \frac{\dot{x}(t)}{x(t)} = g$.

Given the steady state in the capital-to-output ratio, we can move to the steady state in output per worker in our familiar fashion:

$$Y(t)^{1-\alpha} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left([h(t) + m]L(t)\right)^{1-\alpha} \tag{55}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} [h(t) + m]L(t) \tag{56}$$

$$\tag{57}$$

Hence, in steady state

$$Y(t)^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} [h(t) + m]L(t) \tag{58}$$

$$y(t)^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} [h(t) + m]. \tag{59}$$

The equation highlights that trade can be a substitute for skills. By selling some capital goods, the country obtains other capital goods that it would otherwise not have access to. A natural way to think about it is that the country can buy more "unsophisticated" capital goods that it does not know how to produce at home. China is a good example. By foreign countries bringing their technologies with them and using a relatively unskilled Chinese workforce, output was able to grow at tremendous rates.

Note, the Schumpeterian way of thinking may suggest at first that trade may not be useful to increase productivity. After all, the technological leader would have no incentives to buy inferior capital goods from home. However, it becomes obvious that small extensions to the stylized model we have seen change this logic. As discussed above, China, for example, is today the leader in battery technology and could sell these capital goods to the U.S. in exchange of high-precision assembly-line goods. Moreover, one could think about trade in final goods, where China sells final-consumption manufacturing goods to obtain the funds to buy capital goods.

References

López, M. C. and B. de Blas Pérez (2018): "Faraway, so close!, technology diffusion and firm heterogeneity in the medium term cycle of advanced economies," *Documentos de trabajo del Banco de España*, 1–64.